

## WEEK 4: Theoretical Spot Rate Curve and Forward rates. Arbitrage-free pricing of bonds.

### TOPICS:

- Arbitrage-free valuation of bonds
- Theoretical Spot Rate Curve
- Bootstrapping the spot rate curve
- Forward rates
- Deriving forward rates from spot rates

### THE ARBITRAGE-FREE VALUATION APPROACH

The fundamental flaw of the traditional approach is that it views each security as the same package of cash flows. For example, consider a 10-year U.S. Treasury issue with an 8% coupon rate. The cash flows per \$100 of par value would be 19 payments of \$4 every six months and \$104 twenty 6-month periods from now. The traditional practice would discount each cash flow using the same discount rate.

The proper way to view the 10-year 8% coupon Treasury issue is as a package of zero-coupon bonds whose maturity value is equal to the amount of the cash flow and whose maturity date is equal to each cash flow's payment date. Thus, the 10-year 8% coupon Treasury issue should be viewed as 20 zero-coupon bonds. The reason this is the proper way to value a security is that it does not allow arbitrage profit by taking apart or "stripping" a security and selling off the stripped securities at a higher aggregate value than it would cost to purchase the security in the market. We'll illustrate this later. We refer to this approach to valuation as the arbitrage-free valuation approach.

By viewing any financial asset as a package of zero-coupon bonds, a consistent valuation framework can be developed. Viewing a financial asset as a package of zero-coupon bonds means that any two bonds would be viewed as different packages of zero-coupon bonds and valued accordingly.

EXHIBIT 3 Cash Flows for Three 10-Year Hypothetical Treasury Securities Per \$100 of Par Value

Each period is six months

Period	Coupon rate		
	12%	8%	0%
1-19	\$6	\$4	\$0
20	106	104	100

EXHIBIT 4 Comparison of Traditional Approach and Arbitrage-Free Approach in Valuing a Treasury Security

*Each period is six months*

Period	Discount (base interest) rate		Cash flows for*		
	Traditional approach	Arbitrage-free approach	12%	8%	0%
1	10-year Treasury rate	1-period Treasury spot rate	\$6	\$4	\$0
2	10-year Treasury rate	2-period Treasury spot rate	6	4	0
3	10-year Treasury rate	3-period Treasury spot rate	6	4	0
4	10-year Treasury rate	4-period Treasury spot rate	6	4	0
5	10-year Treasury rate	5-period Treasury spot rate	6	4	0
6	10-year Treasury rate	6-period Treasury spot rate	6	4	0
7	10-year Treasury rate	7-period Treasury spot rate	6	4	0
8	10-year Treasury rate	8-period Treasury spot rate	6	4	0
9	10-year Treasury rate	9-period Treasury spot rate	6	4	0
10	10-year Treasury rate	10-period Treasury spot rate	6	4	0
11	10-year Treasury rate	11-period Treasury spot rate	6	4	0
12	10-year Treasury rate	12-period Treasury spot rate	6	4	0
13	10-year Treasury rate	13-period Treasury spot rate	6	4	0
14	10-year Treasury rate	14-period Treasury spot rate	6	4	0
15	10-year Treasury rate	15-period Treasury spot rate	6	4	0
16	10-year Treasury rate	16-period Treasury spot rate	6	4	0
17	10-year Treasury rate	17-period Treasury spot rate	6	4	0
18	10-year Treasury rate	18-period Treasury spot rate	6	4	0
19	10-year Treasury rate	19-period Treasury spot rate	6	4	0
20	10-year Treasury rate	20-period Treasury spot rate	106	104	100

\*Per \$100 of par value.

The difference between the traditional valuation approach and the arbitrage-free approach is illustrated in Exhibit 4, which shows how the three bonds whose cash flows are depicted in Exhibit 3 should be valued. With the traditional approach, the discount rate for all three bonds is the yield on a 10-year U.S. Treasury security. With the arbitrage-free approach, the discount rate for a cash flow is the theoretical rate that the U.S. Treasury would have to pay if it issued a zero-coupon bond with a maturity date equal to the maturity date of the cash flow.

Therefore, to implement the arbitrage-free approach, it is necessary to determine the theoretical rate that the U.S. Treasury would have to pay on a zero-coupon Treasury security for each maturity. As explained in the previous chapter, the name given to the zero-coupon Treasury rate is the Treasury spot rate. In Chapter 6, we will explain how the Treasury spot rate can be calculated. The spot rate for a Treasury security is the interest rate that should be used to discount a default-free cash flow with the same maturity. We call the value of a bond based on spot rates the arbitrage-free value.

EXHIBIT 5 Determination of the Arbitrage-Free Value of an 8% 10-year Treasury

Period	Years	Cash flow (\$)	Spot rate (%)*	Present value (\$) **
1	0.5	4	3.0000	3.9409
2	1.0	4	3.3000	3.8712
3	1.5	4	3.5053	3.7968
4	2.0	4	3.9164	3.7014
5	2.5	4	4.4376	3.5843
6	3.0	4	4.7520	3.4743
7	3.5	4	4.9622	3.3694
8	4.0	4	5.0650	3.2747
9	4.5	4	5.1701	3.1791
10	5.0	4	5.2772	3.0829
11	5.5	4	5.3864	2.9861
12	6.0	4	5.4976	2.8889
13	6.5	4	5.6108	2.7916
14	7.0	4	5.6643	2.7055
15	7.5	4	5.7193	2.6205
16	8.0	4	5.7755	2.5365
17	8.5	4	5.8331	2.4536
18	9.0	4	5.9584	2.3581
19	9.5	4	6.0863	2.2631
20	10.0	104	6.2169	56.3830
			<i>Total</i>	\$115.2621

\*The spot rate is an annual discount rate. The convention to obtain a semiannual discount rate is to take one-half the annual discount rate. So, for period 6 (i.e., 3 years), the spot rate is 4.7520%. The semiannual discount rate is 2.376%.

\*\* The present value for the cash flow is equal to:

$$\frac{\text{Cash flow}}{(1 + \text{Spot rate}/2)^{\text{period}}}$$

### Valuation Using Treasury Spot Rates

For the purposes of our discussion, we will take the Treasury spot rate for each maturity as given. To illustrate how Treasury spot rates are used to compute the arbitrage-free value of a Treasury security, we will use the hypothetical Treasury spot rates shown in the fourth column of Exhibit 5 to value an 8% 10-year Treasury security. The present value of each period's cash flow is shown in the last column. The sum of the present values is the arbitrage-free value for the Treasury security. For the 8% 10-year Treasury, it is \$115.2619.

As a second illustration, suppose that a 4.8% coupon 10-year Treasury bond is being valued based on the Treasury spot rates shown in Exhibit 5. The arbitrage-free value of this bond is \$90.8428 as shown in Exhibit 6.

In the next chapter, we discuss yield measures. The yield to maturity is a measure that would be computed for this bond. We won't show how it is computed in this chapter, but simply state the result. The yield for the 4.8% coupon 10-year Treasury bond is 6.033%.

Notice that the spot rates are used to obtain the price and the price is then used to compute a conventional yield measure. It is important to understand that there are an infinite number of spot rate curves that can generate the same price of \$90.8428 and therefore the same yield.

**EXHIBIT 6** Determination of the Arbitrage-Free Value of a 4.8% 10-year Treasury

Period	Years	Cash flow (\$)	Spot rate (%)*	Present value (\$) **
1	0.5	2.4	3.0000	2.3645
2	1.0	2.4	3.3000	2.3227
3	1.5	2.4	3.5053	2.2781
4	2.0	2.4	3.9164	2.2209
5	2.5	2.4	4.4376	2.1506
6	3.0	2.4	4.7520	2.0846
7	3.5	2.4	4.9622	2.0216
8	4.0	2.4	5.0650	1.9648
9	4.5	2.4	5.1701	1.9075
10	5.0	2.4	5.2772	1.8497
11	5.5	2.4	5.3864	1.7916
12	6.0	2.4	5.4976	1.7334
13	6.5	2.4	5.6108	1.6750
14	7.0	2.4	5.6643	1.6233
15	7.5	2.4	5.7193	1.5723
16	8.0	2.4	5.7755	1.5219
17	8.5	2.4	5.8331	1.4722
18	9.0	2.4	5.9584	1.4149
19	9.5	2.4	6.0863	1.3578
20	10.0	102.4	6.2169	55.5156
			<i>Total</i>	90.8430

\*The spot rate is an annual discount rate. The convention to obtain a semiannual discount rate is to take one-half the annual discount rate. So, for period 6 (i.e., 3 years), the spot rate is 4.7520%. The semiannual discount rate is 2.376%.

\*\* The present value for the cash flow is equal to:

$$\frac{\text{Cash flow}}{(1 + \text{Spot rate}/2)^{\text{period}}}$$

**Reason for Using Treasury Spot Rates**

Thus far, we simply asserted that the value of a Treasury security should be based on discounting each cash flow using the corresponding Treasury spot rate. But what if market participants value a security using the yield for the on-the-run Treasury with a maturity equal to the maturity of the Treasury security being valued? (In other words, what if participants use the yield on coupon-bearing

securities rather than the yield on zero-coupon securities?). Let's see why a Treasury security will have to trade close to its arbitrage-free value.

### **Stripping and the Arbitrage-Free Valuation**

The key in the process is the existence of the Treasury strips market. A dealer has the ability to take apart the cash flows of a Treasury coupon security (i.e., strip the security) and create zero-coupon securities. These zero-coupon securities, which we called Treasury strips, can be sold to investors. At what interest rate or yield can these Treasury strips be sold to investors? They can be sold at the Treasury spot rates. If the market price of a Treasury security is less than its value using the arbitrage-free valuation approach, then a dealer can buy the Treasury security, strip it, and sell off the Treasury strips so as to generate greater proceeds than the cost of purchasing the Treasury security. The resulting profit is an arbitrage profit. Since, as we will see, the value determined by using the Treasury spot rates does not allow for the generation of an arbitrage profit, this is the reason why the approach is referred to as an "arbitrage-free" approach.

To illustrate this, suppose that the yield for the on-the-run 10-year Treasury issue is 6%. (We will see that the Treasury spot rate curve in Exhibit 5 was generated from a yield curve where the on-the-run 10-year Treasury issue was 6%). Suppose that the 8% coupon 10-year Treasury issue is valued using the traditional approach based on 6%. Exhibit 7 shows the value based on discounting all the cash flows at 6% is \$114.8775.

Consider what would happen if the market priced the security at \$114.8775. The value based on the Treasury spot rates (Exhibit 5) is \$115.2621. What can the dealer do? The dealer can buy the 8% 10-year issue for \$114.8775, strip it, and sell the Treasury strips at the spot rates shown in Exhibit 5. By doing so, the proceeds that will be received by the dealer are \$115.2621. This results in an arbitrage profit of \$0.3846 (= \$115.2621 - \$114.8775). Dealers recognizing this arbitrage opportunity will bid up the price of the 8% 10-year Treasury issue in order to acquire it and strip it. At what point will the arbitrage profit disappear? When the security is priced at \$115.2621, the value that we said is the arbitrage-free value.

To understand in more detail where this arbitrage profit is coming from, look at Exhibit 8. The third column shows how much each cash flow can be sold for by the dealer if it is stripped. The values in the third column are simply the present values in Exhibit 5 based on discounting the cash flows at the Treasury spot rates. The fourth column shows how much the dealer is effectively purchasing the cash flow if each cash flow is discounted at 6%. This is the last column in Exhibit 7. The sum of the arbitrage profit from each cash flow stripped is the total arbitrage profit.

EXHIBIT 7 Price of an 8% 10-year Treasury Valued at a 6% Discount Rate

Period	Years	Cash flow (\$)	Spot rate (%)*	Present value (\$) **
1	0.5	4	6.0000	3.8835
2	1.0	4	6.0000	3.7704
3	1.5	4	6.0000	3.6606
4	2.0	4	6.0000	3.5539
5	2.5	4	6.0000	3.4504
6	3.0	4	6.0000	3.3499
7	3.5	4	6.0000	3.2524
8	4.0	4	6.0000	3.1576
9	4.5	4	6.0000	3.0657
10	5.0	4	6.0000	2.9764
11	5.5	4	6.0000	2.8897
12	6.0	4	6.0000	2.8055
13	6.5	4	6.0000	2.7238
14	7.0	4	6.0000	2.6445
15	7.5	4	6.0000	2.5674
16	8.0	4	6.0000	2.4927
17	8.5	4	6.0000	2.4201
18	9.0	4	6.0000	2.3496
19	9.5	4	6.0000	2.2811
20	10.0	104	6.0000	57.5823
			Total	114.8775

\*The discount rate is an annual discount rate. The convention to obtain a semiannual discount rate is to take one-half the annual discount rate. So, since the discount rate for each period is 6%, the semiannual discount rate is 3%.

\*\* The present value for the cash flow is equal to:

$$\frac{\text{Cash flow}}{(1.03)^{\text{period}}}$$

**Reconstitution and Arbitrage-Free Valuation**

We have just demonstrated how coupon stripping of a Treasury issue will force its market value to be close to the value determined by arbitrage-free valuation when the market price is less than the arbitrage-free value. What happens when a Treasury issue's market price is greater than the arbitrage-free value? Obviously, a dealer will not want to strip the Treasury issue since the proceeds generated from stripping will be less than the cost of purchasing the issue.

When such situations occur, the dealer will follow a procedure called reconstitution. Basically, the dealer can purchase a package of Treasury strips so as to create a synthetic (i.e., artificial) Treasury coupon security that is worth more than the same maturity and same coupon Treasury issue.

To illustrate this, consider the 4.8% 10-year Treasury issue whose arbitrage-free value was computed in Exhibit 6. The arbitrage-free value is \$90.8430. Exhibit 9 shows the price assuming the traditional approach where all the cash flows are discounted at a 6% interest rate.

The price is \$91.0735. What the dealer can do is purchase the Treasury strip for each 6-month period at the prices shown in Exhibit 6 and sell short the 4.8% 10-year Treasury coupon issue whose cash flows are being replicated. By doing so, the dealer has the cash flow of a 4.8% coupon 10-year Treasury security at a cost of \$90.8430, thereby generating an arbitrage profit of \$0.2305 (\$91.0735 – \$90.8430). The cash flows from the package of Treasury strips purchased is used to make the payments for the Treasury coupon security shorted. Actually, in practice, this can be done in a more efficient manner using a procedure for reconstitution provided for by the Department of the Treasury.

What forces the market price to the arbitrage-free value of \$90.8430? As dealers sell short the Treasury coupon issue (4.8% 10-year issue), the price of the issue decreases. When the price is driven down to \$90.8430, the arbitrage profit no longer exists.

This process of stripping and reconstitution assures that the price of a Treasury issue will not depart materially from its arbitrage-free value. In other countries, as governments permit the stripping and reconstitution of their issues, the value of non-U.S. government issues have also moved toward their arbitrage-free value.

EXHIBIT 9 Price of a 4.8% 10-Year Treasury Valued at a 6% Discount Rate

Period	Years	Cash flow (\$)	Spot rate (%)	Present value (\$)
1	0.5	2.4	6.0000	2.3301
2	1.0	2.4	6.0000	2.2622
3	1.5	2.4	6.0000	2.1963
4	2.0	2.4	6.0000	2.1324
5	2.5	2.4	6.0000	2.0703
6	3.0	2.4	6.0000	2.0100
7	3.5	2.4	6.0000	1.9514
8	4.0	2.4	6.0000	1.8946
9	4.5	2.4	6.0000	1.8394
10	5.0	2.4	6.0000	1.7858
11	5.5	2.4	6.0000	1.7338
12	6.0	2.4	6.0000	1.6833
13	6.5	2.4	6.0000	1.6343
14	7.0	2.4	6.0000	1.5867
15	7.5	2.4	6.0000	1.5405
16	8.0	2.4	6.0000	1.4956
17	8.5	2.4	6.0000	1.4520
18	9.0	2.4	6.0000	1.4097
19	9.5	2.4	6.0000	1.3687
20	10.0	102.4	6.0000	56.6964
			<i>Total</i>	91.0735

## **SOURCES OF RETURN**

When an investor purchases a fixed income security, he or she can expect to receive a dollar return from one or more of the following sources:

1. the coupon interest payments made by the issuer
2. any capital gain (or capital loss—a negative dollar return) when the security matures, is called, or is sold
3. income from reinvestment of interim cash flows (interest and/or principal payments prior to stated maturity)

Any yield measure that purports to measure the potential return from a fixed income security should consider all three sources of return described above.

### **Coupon Interest Payments**

The most obvious source of return on a bond is the periodic coupon interest payments. For zero-coupon instruments, the return from this source is zero. By purchasing a security below its par value and receiving the full par value at maturity, the investor in a zero-coupon instrument is effectively receiving interest in a lump sum.

### **Capital Gain or Loss**

An investor receives cash when a bond matures, is called, or is sold. If these proceeds are greater than the purchase price, a capital gain results. For a bond held to maturity, there will be a capital gain if the bond is purchased below its par value. For example, a bond purchased for \$94.17 with a par value of \$100 will generate a capital gain of \$5.83 ( $\$100 - \$94.17$ ) if held to maturity. For a callable bond, a capital gain results if the price at which the bond is called (i.e., the call price) is greater than the purchase price. For example, if the bond in our previous example is callable and subsequently called at \$100.5, a capital gain of \$6.33 ( $\$100.50 - \$94.17$ ) will be realized. If the same bond is sold prior to its maturity or before it is called, a capital gain will result if the proceeds exceed the purchase price. So, if our hypothetical bond is sold prior to the maturity date for \$103, the capital gain would be \$8.83 ( $\$103 - \$94.17$ ).

Similarly, for all three outcomes, a capital loss is generated when the proceeds received are less than the purchase price. For a bond held to maturity, there will be a capital loss if the bond is purchased for more than its par value (i.e., purchased at a premium). For example, a bond purchased for \$102.50 with a par value of \$100 will generate a capital loss of \$2.50 ( $\$102.50 - \$100$ ) if held to maturity. For a callable bond, a capital loss results if the price at which the bond is called is less than the purchase price. For example, if the bond in our example is callable and subsequently called at \$100.50, a capital loss of \$2 ( $\$102.50 - \$100.50$ ) will be realized. If the same bond is sold prior to its maturity or before it is called, a capital loss will result if the sale price is less than the purchase price.

So, if our hypothetical bond is sold prior to the maturity date for \$98.50, the capital loss would be \$4 (\$102.50 – \$98.50).

### **Reinvestment Income**

Prior to maturity, with the exception of zero-coupon instruments, fixed income securities make periodic interest payments that can be reinvested. Amortizing securities (such as mortgage-backed securities and asset-backed securities) make periodic principal payments that can be reinvested prior to final maturity. The interest earned from reinvesting the interim cash flows (interest and/or principal payments) prior to final or stated maturity is called reinvestment income.

### **TRADITIONAL YIELD MEASURES**

Yield measures cited in the bond market include current yield, yield to maturity, yield to call, yield to put, yield to worst, and cash flow yield. These yield measures are expressed as a percent return rather than a dollar return. Below we explain how each measure is calculated and its limitations.

#### **Current Yield**

The current yield relates the annual dollar coupon interest to a bond's market price. The formula for the current yield is:

**Current yield = annual dollar coupon interest / price**

For example, the current yield for a 7% 8-year bond whose price is \$94.17 is 7.43% as shown below:

Annual dollar coupon interest =  $0.07 \times \$100 = \$7$

Price = \$94.17

Current yield =  $\$7 / \$94.17 = 0.0743$  or 7.43%

The current yield will be greater than the coupon rate when the bond sells at a discount; the reverse is true for a bond selling at a premium. For a bond selling at par, the current yield will be equal to the coupon rate.

The drawback of the current yield is that it considers only the coupon interest and no other source for an investor's return. No consideration is given to the capital gain an investor will realize when a bond purchased at a discount is held to maturity; nor is there any recognition of the capital loss an investor will realize if a bond purchased at a premium is held to maturity. No consideration is given to reinvestment income.

## Yield to Maturity

The most popular measure of yield in the bond market is the yield to maturity. The yield to maturity is the interest rate that will make the present value of a bond's cash flows equal to its market price plus accrued interest. To find the yield to maturity, we first determine the expected cash flows and then search, by trial and error, for the interest rate that will make the present value of cash flows equal to the market price plus accrued interest. (This is simply a special case of an internal rate of return (IRR) calculation where the cash flows are those received if the bond is held to the maturity date.) In the illustrations presented in this chapter, we assume that the next coupon payment will be six months from now so that there is no accrued interest.

To illustrate, consider a 7% 8-year bond selling for \$94.17. The cash flows for this bond are (1) 16 payments every 6-months of \$3.50 and (2) a payment sixteen 6-month periods from now of \$100. The present value using various semi-annual discount (interest) rates is:

Semiannual interest rate	3.5%	3.6%	3.7%	3.8%	3.9%	4.0%
Present value	100.00	98.80	97.62	96.45	95.30	94.17

When a 4.0% interest rate is used, the present value of the cash flows is equal to \$94.17, which is the price of the bond. Hence, 4.0% is the semi-annual yield to maturity. The market convention adopted to annualize the semi-annual yield to maturity is to double it and call that the yield to maturity. Thus, the yield to maturity for the above bond is 8% (2 times 4.0%). The yield to maturity computed using this convention—doubling the semi-annual yield—is called a bond-equivalent yield.

The following relationships between the price of a bond, coupon rate, current yield, and yield to maturity hold:

Bond selling at	Relationship
par	coupon rate = current yield = yield to maturity
discount	coupon rate < current yield < yield to maturity
premium	coupon rate > current yield > yield to maturity

## The Bond-Equivalent Yield Convention

The convention developed in the bond market to move from a semi-annual yield to an annual yield is to simply double the semi-annual yield. As just noted, this is called the bond-equivalent yield. In general, when one doubles a semi-annual yield (or a semi-annual return) to obtain an annual measure, one is said to be computing the measure on a bond-equivalent basis.

Students of the bond market are troubled by this convention. The two questions most commonly asked are: First, why is the practice of simply doubling a semi-annual yield followed? Second, wouldn't it be more appropriate to compute the effective annual yield by compounding the semi-annual yield?

The answer to the first question is that it is simply a convention. There is no danger with a convention unless you use it improperly. The fact is that market participants recognize that a yield (or return) is computed on a semi-annual basis by convention and adjust accordingly when using the number. So, if the bond-equivalent yield on a security purchased by an investor is 6%, the investor knows the semi-annual yield is 3%. Given that, the investor can use that semi-annual yield to compute an effective annual yield or any other annualized measure desired.

For a manager comparing the yield on a security as an asset purchased to a yield required on a liability to satisfy, the yield figure will be measured in a manner consistent with that of the yield required on the liability.

The answer to the second question is that it is true that computing an effective annual yield would be better. But so what? Once we discover the limitations of yield measures in general, we will question whether or not an investor should use a bond-equivalent yield measure or an effective annual yield measure in making investment decisions. That is, when we identify the major problems with yield measures, the doubling of a semi-annual yield is the least of our problems.

So, don't lose any sleep over this convention. Just make sure that you use a bond-equivalent yield measure properly.

### **Limitations of Yield-to-Maturity Measure**

The yield to maturity considers not only the coupon income but any capital gain or loss that the investor will realize by holding the bond to maturity. The yield to maturity also considers the timing of the cash flows. It does consider reinvestment income; however, it assumes that the coupon payments can be reinvested at an interest rate equal to the yield to maturity. So, if the yield to maturity for a bond is 8%, for example, to earn that yield the coupon payments must be reinvested at an interest rate equal to 8%.

The illustrations below clearly demonstrate this. In the illustrations, the analysis will be in terms of dollars. Be sure you keep in mind the difference between the total future dollars, which is equal to all the dollars an investor expects to receive (including the recovery of the principal), and the total dollar return, which is equal to the dollars an investor expects to realize from the three sources of return (coupon payments, capital gain/loss, and reinvestment income).

Suppose an investor has \$94.17 and places the funds in a certificate of deposit (CD) that matures in 8 years. Let's suppose that the bank agrees to pay 4% interest every six months.

This means that the bank is agreeing to pay 8% on a bond equivalent basis (i.e., doubling the semi-annual yield). We can translate all of this into the total future dollars that will be generated by this investment at the end of 8 years. From the standard formula for the future value of an investment today, we can determine the total future dollars as:

$$\$94.17 \times (1.04)^{16} = \$176.38$$

So, to an investor who invests \$94.17 for 8 years at an 8% yield on a bond equivalent basis and interest is paid semi-annually, the investment will generate \$176.38. Decomposing the total future dollars we see that:

Total future dollars = \$176.38

Return of principal = \$94.17

Total interest from CD = \$82.21

Thus, any investment that promises a yield of 8% on a bond equivalent basis for 8 years on an investment of \$94.17 must generate total future dollars of \$176.38 or equivalently a return from all sources of \$82.21. That is, if we look at the three sources of a bond return that offered an 8% yield with semi-annual coupon payments and sold at a price of \$94.17, the following would have to hold:

Coupon interest

+ Capital gain

+ Reinvestment income

= Total dollar return = Total interest from CD = \$82.21

Now, instead of a certificate of deposit, suppose that an investor purchases a bond with a coupon rate of 7% that matures in 8 years. We know that the three sources of return are coupon income, capital gain/loss, and reinvestment income. Suppose that the price of this bond is \$94.17. The yield to maturity for this bond (on a bond equivalent basis) is 8%. Notice that this is the same type of investment as the certificate of deposit—the bank offered an 8% yield on a bond equivalent basis for 8 years and made payments semi-annually. So, what should the investor in this bond expect in terms of total future dollars? As we just demonstrated, an investment of \$94.17 must generate \$176.38 in order to say that it provided a yield of 8%.

Or equivalently, the total dollar return that must be generated is \$82.21. Let's look at what in fact is generated in terms of dollar return.

The coupon is \$3.50 every six months. So the dollar return from the coupon interest is \$3.50 for 16 six-month periods, or \$56. When the bond matures, there is a capital gain of \$5.83 (\$100 – \$94.17). Therefore, based on these two sources of return we have:

Coupon interest = \$56.00

Capital gain = \$5.83

Dollar return without reinvestment income = \$61.83

Something's wrong here. Only \$61.83 is generated from the bond whereas \$82.21 is needed in order to say that this bond provided an 8% yield. That is, there is a dollar return shortfall of \$20.38 (\$82.21 – \$61.83). How is this dollar return shortfall generated?

Recall that in the case of the certificate of deposit, the bank does the reinvesting of the principal and interest, and pays 4% every six months or 8% on a bond equivalent basis. In contrast, for the bond, the investor has to reinvest any coupon interest until the bond matures.

It is the reinvestment income that must generate the dollar return shortfall of \$20.38. But at what yield will the investor have to reinvest the coupon payments in order to generate the \$20.38? The answer is: the yield to maturity.<sup>2</sup> That is, the reinvestment income will be \$20.38 if each semi-annual coupon payment of \$3.50 can be reinvested at a semi-annual yield of 4% (one half the yield to maturity). The reinvestment income earned on a given coupon payment of \$3.50, if it is invested from the time of receipt in period  $t$  to the maturity date (16 periods in our example) at a 4% semi-annual rate, is:

$$\$3.50 (1.04)^{16-t} - \$3.50$$

The first coupon payment ( $t = 1$ ) can be reinvested for 15 periods. Applying the formula above we find the reinvestment income earned on the first coupon payment is:

$$\$3.50 (1.04)^{16-1} - \$3.50 = \$2.80$$

Similarly, the reinvestment income for all coupon payments is shown below:

Period	Periods reinvested	Coupon payment	Reinvestment income
1	15	\$3.5	\$2.80
2	14	3.5	2.56
3	13	3.5	2.33
4	12	3.5	2.10
5	11	3.5	1.89
6	10	3.5	1.68
7	9	3.5	1.48
8	8	3.5	1.29
9	7	3.5	1.11
10	6	3.5	0.93
11	5	3.5	0.76
12	4	3.5	0.59
13	3	3.5	0.44
14	2	3.5	0.29
15	1	3.5	0.14
16	0	3.5	0.00
		Total	\$20.39

<sup>2</sup>This can be verified by using the future value of an annuity. The future of an annuity is given by the following formula:

$$\text{Annuity payment} = \left[ \frac{(1+i)^n - 1}{i} \right]$$

where  $i$  is the interest rate and  $n$  is the number of periods.

In our example,  $i$  is 4%,  $n$  is 16, and the amount of the annuity is the semiannual coupon of \$3.50. Therefore, the future value of the coupon payment is

$$\$3.50 \left[ \frac{(1.04)^{16} - 1}{0.04} \right] = \$76.38$$

Since the coupon payments are \$56, the reinvestment income is \$20.38 (\$76.38 - \$56). This is the amount that is necessary to produce the dollar return shortfall in our example.

The total reinvestment income is \$20.39 (differing from \$20.38 due to rounding). So, with the reinvestment income of \$20.38 at 4% semi-annually (i.e., one half the yield to maturity on a bond-equivalent basis), the total dollar return is

Coupon interest = \$56.00

Capital gain = \$5.83

Reinvestment income = \$20.38

Total dollar return = \$82.21

In our illustration, we used an investment in a certificate of deposit to show what the total future dollars will have to be in order to obtain a yield of 8% on an investment of \$94.17 for 8 years when interest payments are semi-annual. However, this holds for any type of investment, not just a certificate of deposit. For example, if an investor is told that he or she can purchase a debt

instrument for \$94.17 that offers an 8% yield (on a bond-equivalent basis) for 8 years and makes interest payments semi-annually, then the investor should translate this yield into the following:

I should be receiving total future dollars of \$176.38

I should be receiving a total dollar return of \$82.21

It is always important to think in terms of dollars (or pound sterling, yen, or other currency) because “yield measures” are misleading.

We can also see that the reinvestment income can be a significant portion of the total dollar return. In our example, the total dollar return is \$82.21 and the total dollar return from reinvestment income to make up the shortfall is \$20.38. This means that reinvestment income is about 25% of the total dollar return.

This is such an important point that we should go through this one more time for another bond. Suppose an investor purchases a 15-year 8% coupon bond at par value (\$100). The yield for this bond is simple to determine since the bond is trading at par. The yield is equal to the coupon rate, 8%. Let’s translate this into dollars. We know that if an investor makes an investment of \$100 for 15 years that offers an 8% yield and the interest payments are semi-annual, the total future dollars will be:

$$\$100 \times (1.04)^{30} = \$324.34$$

Decomposing the total future dollars we see that:

Total future dollars = \$324.34

Return of principal = \$100.00

Total dollar return = \$224.34

Without reinvestment income, the dollar return is:

Coupon interest = \$120

Capital gain = \$0

Dollar return without reinvestment income = \$120

Note that the capital gain is \$0 because the bond is purchased at par value. The dollar return shortfall is therefore \$104.34 (\$224.34 – \$120). This shortfall is made up if the coupon payments can be reinvested at a yield of 8% (the yield on the bond at the time of purchase). For this bond, the reinvestment income is 46.5% of the total dollar return needed to produce a yield of 8% (\$104.34/\$224.34).

Clearly, the investor will only realize the yield to maturity stated at the time of purchase if the following two assumptions hold:

***Assumption 1: the coupon payments can be reinvested at the yield to maturity***

***Assumption 2: the bond is held to maturity***

With respect to the first assumption, the risk that an investor faces is that future interest rates will be less than the yield to maturity at the time the bond is purchased, known as reinvestment risk. If the bond is not held to maturity, the investor faces the risk that he may have to sell for less than the purchase price, resulting in a return that is less than the yield to maturity, known as interest rate risk.

### **Factors Affecting Reinvestment Risk**

There are two characteristics of a bond that affect the degree of reinvestment risk:

***Characteristic 1.*** For a given yield to maturity and a given non-zero coupon rate, the longer the maturity, the more the bond's total dollar return depends on reinvestment income to realize the yield to maturity at the time of purchase. That is, the greater the reinvestment risk.

The implication is the yield to maturity measure for long-term maturity coupon bonds tells little about the potential return that an investor may realize if the bond is held to maturity.

For long-term bonds, in high interest rate environments, the reinvestment income component may be as high as 70% of the bond's total dollar return.

***Characteristic 2.*** For a coupon paying bond, for a given maturity and a given yield to maturity, the higher the coupon rate, the more dependent the bond's total dollar return will be on the reinvestment of the coupon payments in order to produce the yield to maturity at the time of purchase.

This means that holding maturity and yield to maturity constant, bonds selling at a premium will be more dependent on reinvestment income than bonds selling at par. This is because the reinvestment income has to make up the capital loss due to amortizing the price premium when holding the bond to maturity. In contrast, a bond selling at a discount will be less dependent on reinvestment income than a bond selling at par because a portion of the return is coming from the capital gain due to accrediting the price discount when holding the bond to maturity. For zero-coupon bonds, none of the bond's total dollar return is dependent on reinvestment income. So, a zero-coupon bond has no reinvestment risk if held to maturity.

The dependence of the total dollar return on reinvestment income for bonds with different coupon rates and maturities is shown in Exhibit 1.

**EXHIBIT 1** Percentage of Total Dollar Return from Reinvestment Income for a Bond to Generate an 8% Yield (BEY)

	Years to maturity				
	2	3	5	8	15
<b>Bond with a 7% coupon</b>					
Price	98.19	97.38	95.94	94.17	91.35
% of total	5.2%	8.6%	15.2%	24.8%	44.5%
<b>Bond with an 8% coupon</b>					
Price	100.00	100.00	100.00	100.00	100.00
% of total	5.8%	9.5%	16.7%	26.7%	46.5%
<b>Bond with a 12% coupon</b>					
Price	107.26	110.48	116.22	122.30	134.58
% of total	8.1%	12.9%	21.6%	31.0%	51.8%

**THEORETICAL SPOT RATES**

The theoretical spot rates for Treasury securities represent the appropriate set of interest rates that should be used to value default-free cash flows. A default-free theoretical spot rate curve can be constructed from the observed Treasury yield curve. There are several approaches that are used in practice. The approach that we describe below for creating a theoretical spot rate curve is called bootstrapping. (The bootstrapping method described here is also used in constructing a theoretical spot rate curve for LIBOR.)

**Bootstrapping**

Bootstrapping begins with the yield for the on-the-run Treasury issues because there is no credit risk and no liquidity risk. In practice, however, there is a problem of obtaining a sufficient number of data points for constructing the U.S. Treasury yield curve. In the United States, the U.S. Department of the Treasury currently issues 3-month and 6-month Treasury bills and 2-year, 5-year, and 10-year Treasury notes. Treasury bills are zero-coupon instruments and Treasury notes are coupon-paying instruments. Hence, there are not many data points from which to construct a Treasury yield curve, particularly after two years. At one time, the U.S. Treasury issued 30-year Treasury bonds. Since the Treasury no longer issues 30-year bonds, market participants currently use the last issued Treasury bond (which has a maturity less than 30 years) to estimate the 30-year yield. The 2-year, 5-year, and 10-year Treasury notes and an estimate of the 30-year Treasury bond are used to construct the Treasury yield curve.

On September 5, 2003, Lehman Brothers reported the following values for these four yields:

2 year 1.71%

5 year 3.25%

10 year 4.35%

30 year 5.21%

To fill in the yield for the 25 missing whole year maturities (3 year, 4 year, 6 year, 7 year, 8 year, 9 year, 11 year, and so on to the 29-year maturity), the yield for the 25 whole year maturities are interpolated from the yield on the surrounding maturities. The simplest interpolation, and the one most commonly used in practice, is simple **linear interpolation**.

For example, suppose that we want to fill in the gap for each one year of maturity. To determine the amount to add to the on-the-run Treasury yield as we go from the lower maturity to the higher maturity, the following formula is used:

***[Yield at higher maturity – Yield at lower maturity] / Number of years between two observed maturity points***

The estimated on-the-run yield for all intermediate whole-year maturities is found by adding the amount computed from the above formula to the yield at the lower maturity.

For example, using the September 5, 2003 yields, the 5-year yield of 3.25% and the 10-year yield of 4.35% are used to obtain the interpolated 6-year, 7-year, 8-year, and 9-year yields by first calculating:

$$(4.35\% - 3.25\%) / 5 = 0.22\%$$

$$\text{Then, interpolated 6-year yield} = 3.25\% + 0.22\% = 3.47\%$$

$$\text{interpolated 7-year yield} = 3.47\% + 0.22\% = 3.69\%$$

$$\text{interpolated 8-year yield} = 3.69\% + 0.22\% = 3.91\%$$

$$\text{interpolated 9-year yield} = 3.91\% + 0.22\% = 4.13\%$$

Thus, when market participants talk about a yield on the Treasury yield curve that is not one of the on-the-run maturities—for example, the 8-year yield—it is only an approximation. Notice that there is a large gap between maturity points. This may result in misleading yields for the interim maturity points when estimated using the linear interpolation method, a point that we return to later in this chapter.

To illustrate bootstrapping, we will use the Treasury yields shown in Exhibit 4 for maturities up to 10 years using 6-month periods.<sup>7</sup> Thus, there are 20 Treasury yields shown.

The yields shown are assumed to have been interpolated from the on-the-run Treasury issues. Exhibit 5 shows the Treasury yield curve based on the yields shown in Exhibit 4. Our objective is to show how the values in the last column of Exhibit 4 (labelled “Spot rate”) are obtained.

Throughout the analysis and illustrations to come, it is important to remember that the basic principle is the value of the Treasury coupon security should be equal to the value of the package of zero-coupon Treasury securities that duplicates the coupon bond’s cash flows. We saw this in Chapter 5 when we discussed arbitrage-free valuation.

Consider the 6-month and 1-year Treasury securities in Exhibit 4. These two securities are called treasury bills and they are issued as zero-coupon instruments. Therefore, the annualized yield (not the discount yield) of 3.00% for the 6-month Treasury security is equal to the 6-month spot rate.<sup>8</sup> Similarly, for the 1-year Treasury security, the cited yield of 3.30% is the 1-year spot rate. Given these two spot rates, we can compute the spot rate for a theoretical 1.5-year zero-coupon Treasury.

The value of a theoretical 1.5-year Treasury should equal the present value of the three cash flows from the 1.5-year coupon Treasury, where the yield used for discounting is the spot rate corresponding to the time of receipt of each six-month cash flow.

**EXHIBIT 4** Hypothetical Treasury Yields (Interpolated)

Period	Years	Annual par yield to maturity (BEY) (%)*	Price	Spot rate (BEY) (%)*
1	0.5	3.00	—	3.0000
2	1.0	3.30	—	3.3000
3	1.5	3.50	100.00	3.5053
4	2.0	3.90	100.00	3.9164
5	2.5	4.40	100.00	4.4376
6	3.0	4.70	100.00	4.7520
7	3.5	4.90	100.00	4.9622
8	4.0	5.00	100.00	5.0650
9	4.5	5.10	100.00	5.1701
10	5.0	5.20	100.00	5.2772
11	5.5	5.30	100.00	5.3864
12	6.0	5.40	100.00	5.4976
13	6.5	5.50	100.00	5.6108
14	7.0	5.55	100.00	5.6643
15	7.5	5.60	100.00	5.7193
16	8.0	5.65	100.00	5.7755
17	8.5	5.70	100.00	5.8331
18	9.0	5.80	100.00	5.9584
19	9.5	5.90	100.00	6.0863
20	10.0	6.00	100.00	6.2169

\*The yield to maturity and the spot rate are annual rates. They are reported as bond-equivalent yields. To obtain the semiannual yield or rate, one half the annual yield or annual rate is used.

EXHIBIT 5 Treasury Par Yield Curve

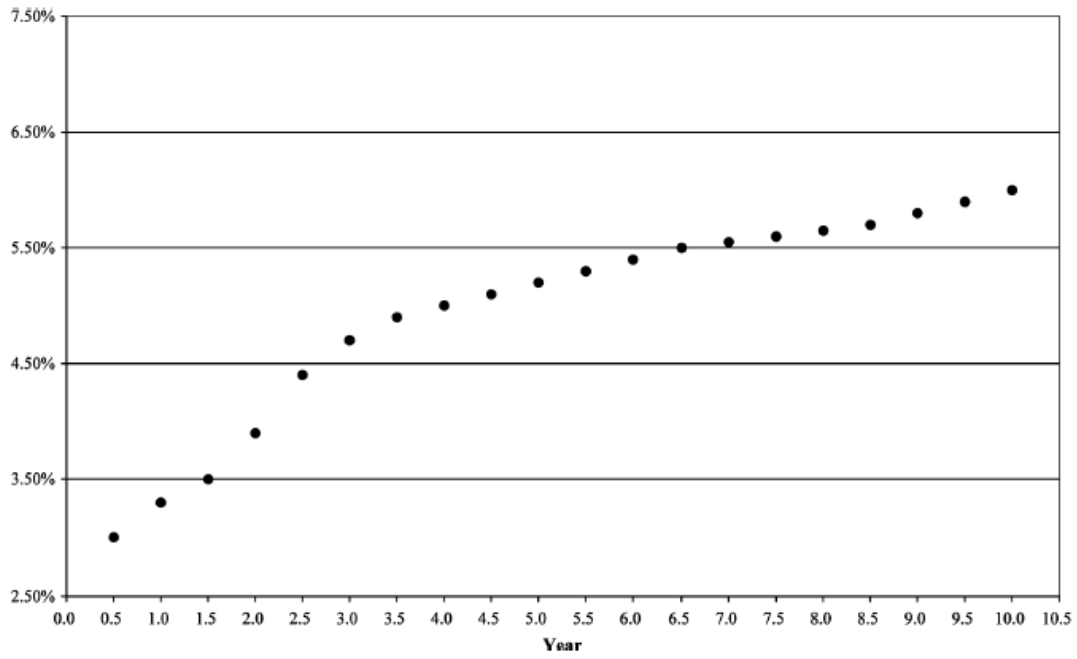
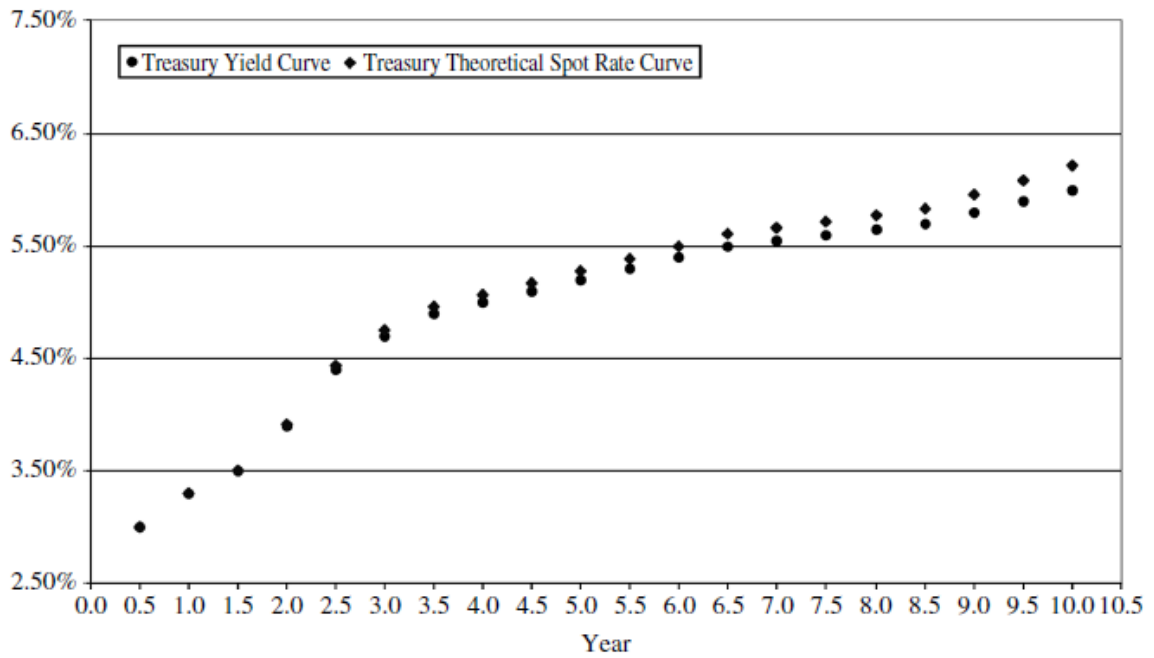


EXHIBIT 6 Theoretical Spot Rate Curve and Treasury Yield Curve



Since all the coupon bonds are selling at par, as explained in the previous section, the yield to maturity for each bond is the coupon rate. Using \$100 par, the cash flows for the 1.5-year coupon Treasury are:

$$0.5 \text{ year } 0.035 \times \$100 \times 0.5 = \$1.75$$

$$1.0 \text{ year } 0.035 \times \$100 \times 0.5 = \$1.75$$

$$1.5 \text{ years } 0.035 \times \$100 \times 0.5 + 100 = \$101.75$$

The present value of the cash flows is then:

$$\frac{1.75}{(1 + z_1)^1} + \frac{1.75}{(1 + z_2)^2} + \frac{101.75}{(1 + z_3)^3}$$

where

$z_1$  = one-half the annualized 6-month theoretical spot rate

$z_2$  = one-half the 1-year theoretical spot rate

$z_3$  = one-half the 1.5-year theoretical spot rate

Since the 6-month spot rate is 3% and the 1-year spot rate is 3.30%, we know that:

$$z_1 = 0.0150 \text{ and } z_2 = 0.0165$$

We can compute the present value of the 1.5-year coupon Treasury security as:

$$\frac{1.75}{(1+z_1)^1} + \frac{1.75}{(1+z_2)^2} + \frac{101.75}{(1+z_3)^3} = \frac{1.75}{(1.015)^1} + \frac{1.75}{(1.0165)^2} + \frac{101.75}{(1+z_3)^3}$$

Since the price of the 1.5-year coupon Treasury security is par value (see Exhibit 4), the following relationship must hold:<sup>9</sup>

$$\frac{1.75}{(1.015)^1} + \frac{1.75}{(1.0165)^2} + \frac{101.75}{(1+z_3)^3} = 100$$

We can solve for the theoretical 1.5-year spot rate as follows:

$$\begin{aligned} 1.7241 + 1.6936 + \frac{101.75}{(1+z_3)^3} &= 100 \\ \frac{101.75}{(1+z_3)^3} &= 96.5822 \\ (1+z_3)^3 &= \frac{101.75}{96.5822} \\ z_3 &= 0.0175265 = 1.7527\% \end{aligned}$$

Doubling this yield, we obtain the bond-equivalent yield of 3.5053%, which is the theoretical 1.5-year spot rate. That rate is the rate that the market would apply to a 1.5-year zero-coupon Treasury security if, in fact, such a security existed. In other words, all Treasury cash flows to be received 1.5 years from now should be valued (i.e., discounted) at 3.5053%.

Given the theoretical 1.5-year spot rate, we can obtain the theoretical 2-year spot rate. The cash flows for the 2-year coupon Treasury in Exhibit 3 are:

0.5 year	$0.039 \times \$100 \times 0.5$	=	\$1.95
1.0 year	$0.039 \times \$100 \times 0.5$	=	\$1.95
1.5 years	$0.039 \times \$100 \times 0.5$	=	\$1.95
2.0 years	$0.039 \times \$100 \times 0.5 + 100$	=	\$101.95

The present value of the cash flows is then:

$$\frac{1.95}{(1+z_1)^1} + \frac{1.95}{(1+z_2)^2} + \frac{1.95}{(1+z_3)^3} + \frac{101.95}{(1+z_4)^4}$$

where  $z_4$  = one-half the 2-year theoretical spot rate.

Since the 6-month spot rate, 1-year spot rate, and 1.5-year spot rate are 3.00%, 3.30%, and 3.5053%, respectively, then:

$$z_1 = 0.0150 \quad z_2 = 0.0165 \quad z_3 = 0.017527$$

Therefore, the present value of the 2-year coupon Treasury security is:

$$\frac{1.95}{(1.0150)^1} + \frac{1.95}{(1.0165)^2} + \frac{1.95}{(1.017527)^3} + \frac{101.95}{(1 + z_4)^4}$$

Since the price of the 2-year coupon Treasury security is par, the following relationship must hold:

$$\frac{1.95}{(1.0150)^1} + \frac{1.95}{(1.0165)^2} + \frac{1.95}{(1.017527)^3} + \frac{101.95}{(1 + z_4)^4} = 100$$

We can solve for the theoretical 2-year spot rate as follows:

$$\begin{aligned} \frac{101.95}{(1 + z_4)^4} &= 94.3407 \\ (1 + z_4)^4 &= \frac{101.95}{94.3407} \\ z_4 &= 0.019582 = 1.9582\% \end{aligned}$$

Doubling this yield, we obtain the theoretical 2-year spot rate bond-equivalent yield of 3.9164%. One can follow this approach sequentially to derive the theoretical 2.5-year spot rate from the calculated values of  $z_1$ ,  $z_2$ ,  $z_3$ , and  $z_4$  (the 6-month-, 1-year-, 1.5-year-, and 2-year rates), and the price and coupon of the 2.5-year bond in Exhibit 4. Further, one could derive theoretical spot rates for the remaining 15 half-yearly rates.

The spot rates thus obtained are shown in the last column of Exhibit 4. They represent the term structure of default-free spot rate for maturities up to 10 years at the particular time to which the bond price quotations refer. In fact, it is the default-free spot rates shown in Exhibit 4 that were used in our illustrations in the previous chapter.

Exhibit 6 shows a plot of the spot rates. The graph is called the theoretical spot rate curve. Also shown on Exhibit 6 is a plot of the par yield curve from Exhibit 5. Notice that the theoretical spot rate curve lies above the par yield curve. This will always be the case when the par yield curve is upward sloping. When the par yield curve is downward sloping, the theoretical spot rate curve will lie below the par yield curve.

## FORWARD RATES

We have seen how a default-free theoretical spot rate curve can be extrapolated from the Treasury yield curve. Additional information useful to market participants can be extrapolated from the

default-free theoretical spot rate curve: forward rates. Under certain assumptions described later, these rates can be viewed as the market's consensus of future interest rates.

Examples of forward rates that can be calculated from the default-free theoretical spot rate curve are the:

- 6-month forward rate six months from now
- 6-month forward rate three years from now
- 1-year forward rate one year from now
- 3-year forward rate two years from now
- 5-year forward rates three years from now

Since the forward rates are implicitly extrapolated from the default-free theoretical spot rate curve, these rates are sometimes referred to as implied forward rates. We begin by showing how to compute the 6-month forward rates. Then we explain how to compute any forward rate.

While we continue to use the Treasury yield curve in our illustrations, as noted earlier, a LIBOR spot rate curve can also be constructed using the bootstrapping methodology and forward rates for LIBOR can be obtained in the same manner as described below.

### **Deriving 6-Month Forward Rates**

To illustrate the process of extrapolating 6-month forward rates, we will use the yield curve and corresponding spot rate curve from Exhibit 4. We will use a very simple arbitrage principle as we did earlier in this chapter to derive the spot rates. Specifically, if two investments have the same cash flows and have the same risk, they should have the same value.

Consider an investor who has a 1-year investment horizon and is faced with the following two alternatives:

- buy a 1-year Treasury bill, or
- buy a 6-month Treasury bill and, when it matures in six months, buy another 6-month Treasury bill.

The investor will be indifferent toward the two alternatives if they produce the same return over the 1-year investment horizon. The investor knows the spot rate on the 6-month Treasury bill and the 1-year Treasury bill. However, he does not know what yield will be on a 6-month Treasury bill purchased six months from now. That is, he does not know the 6-month forward rate six months from now. Given the spot rates for the 6-month Treasury bill and the 1-year Treasury bill, the forward rate on a 6-month Treasury bill is the rate that equalizes the dollar return between the two alternatives.

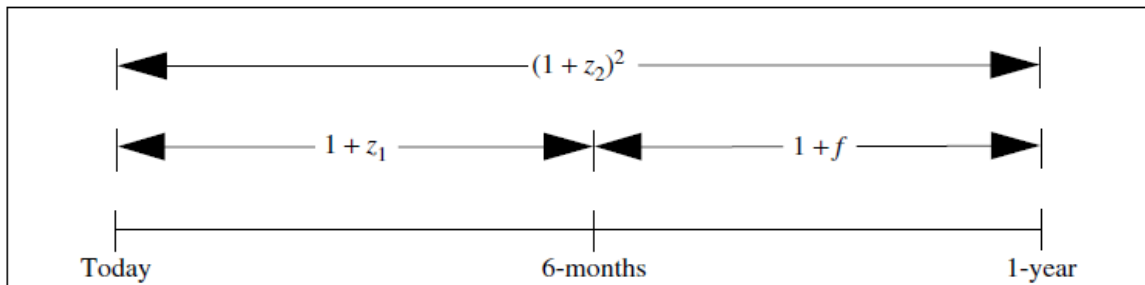
To see how that rate can be determined, suppose that an investor purchased a 6-month Treasury bill for \$X. At the end of six months, the value of this investment would be:

$$X(1 + z_1)$$

where  $z_1$  is one-half the bond-equivalent yield (BEY) of the theoretical 6-month spot rate. Let  $f$  represent one-half the forward rate (expressed as a BEY) on a 6-month Treasury bill available six months from now. If the investor were to rollover his investment by purchasing that bill at that time, then the future dollars available at the end of one year from the X investment would be:

$$X(1 + z_1)(1 + f)$$

EXHIBIT 11 Graphical Depiction of the Six-Month Forward Rate Six Months from Now



Now consider the alternative of investing in a 1-year Treasury bill. If we let  $z_2$  represent one-half the BEY of the theoretical 1-year spot rate, then the future dollars available at the end of one year from the X investment would be:

$$X(1 + z_2)^2$$

The reason that the squared term appears is that the amount invested is being compounded for two periods. (Recall that each period is six months.) The two choices are depicted in Exhibit 11. Now we are prepared to analyze the investor's choices and what this says about forward rates. The investor will be indifferent toward the two alternatives confronting him if he makes the same dollar investment (\$X) and receives the same future dollars from both alternatives at the end of one year. That is, the investor will be indifferent if:

$$X(1 + z_1)(1 + f) = X(1 + z_2)^2$$

Solving for  $f$ , we get:

$$f = \frac{(1 + z_2)^2}{(1 + z_1)} - 1$$

Doubling  $f$  gives the BEY for the 6-month forward rate six months from now. We can illustrate the use of this formula with the theoretical spot rates shown in Exhibit 4. From that exhibit, we know that:

6-month bill spot rate = 0.030, therefore  $z_1 = 0.0150$

1-year bill spot rate = 0.033, therefore  $z_2 = 0.0165$

Substituting into the formula, we have:

$$f = \frac{(1.0165)^2}{(1.0150)} - 1 = 0.0180 = 1.8\%$$

Therefore, the 6-month forward rate six months from now is 3.6% ( $1.8\% \times 2$ ) BEY. Let's confirm our results. If  $X$  is invested in the 6-month Treasury bill at 1.5% and the proceeds then reinvested for six months at the 6-month forward rate of 1.8%, the total proceeds from this alternative would be:

$$X(1.015)(1.018) = 1.03327 X$$

Investment of  $X$  in the 1-year Treasury bill at one-half the 1-year rate, 1.0165%, would produce the following proceeds at the end of one year:

$$X(1.0165)^2 = 1.03327 X$$

Both alternatives have the same payoff if the 6-month Treasury bill yield six months from now is 1.8% (3.6% on a BEY). This means that, if an investor is guaranteed a 1.8% yield (3.6% BEY) on a 6-month Treasury bill six months from now, he will be indifferent toward the two alternatives.

The same line of reasoning can be used to obtain the 6-month forward rate beginning at any time period in the future. For example, the following can be determined:

- the 6-month forward rate three years from now
- the 6-month forward rate five years from now

The notation that we use to indicate 6-month forward rates is  ${}_1f_m$  where the subscript 1 indicates a 1-period (6-month) rate and the subscript  $m$  indicates the period beginning  $m$  periods from now. When  $m$  is equal to zero, this means the current rate. Thus, the first 6-month forward rate is simply the current 6-month spot rate. That is,  ${}_1f_0 = z_1$ .

The general formula for determining a 6-month forward rate is:

$${}_1f_m = \frac{(1 + z_{m+1})^{m+1}}{(1 + z_m)^m} - 1$$

For example, suppose that the 6-month forward rate four years (eight 6-month periods) from now is sought. In terms of our notation,  $m$  is 8 and we seek  ${}_1f_8$ . The formula is then:

$${}_1f_8 = \frac{(1 + z_9)^9}{(1 + z_8)^8} - 1$$

From Exhibit 4, since the 4-year spot rate is 5.065% and the 4.5-year spot rate is 5.1701%,  $z_8$  is 2.5325% and  $z_9$  is 2.58505%. Then,

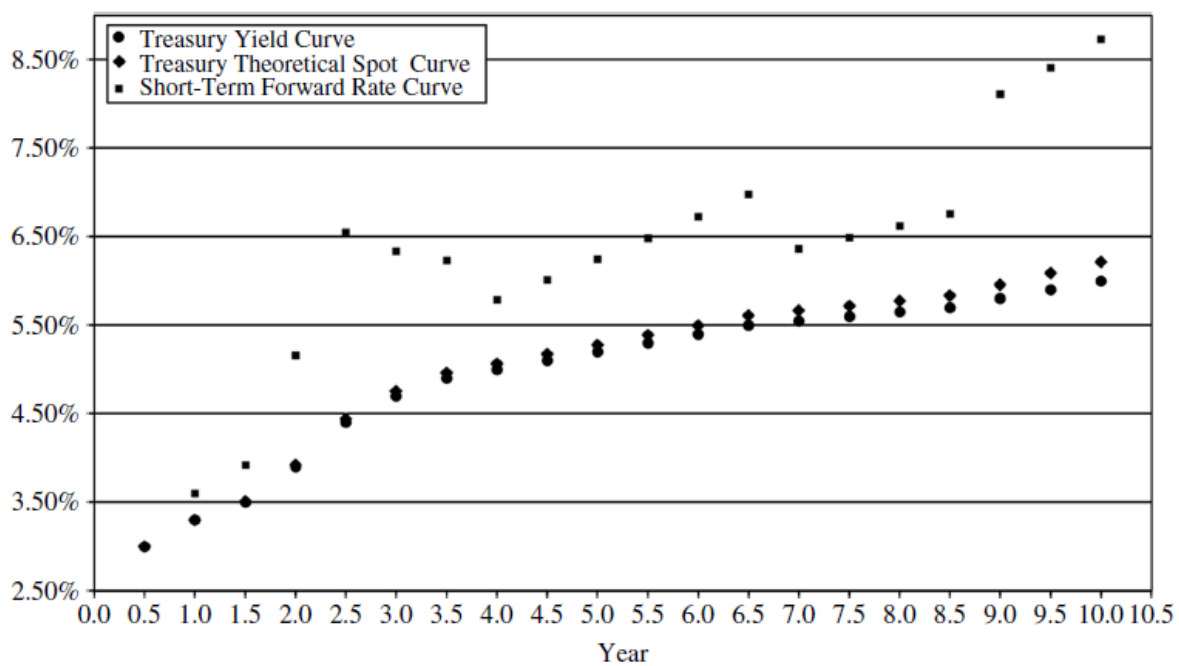
$${}_1f_8 = \frac{(1.0258505)^9}{(1.025325)^8} - 1 = 3.0064\%$$

Doubling this rate gives a 6-month forward rate four years from now of 6.01% Exhibit 12 shows all of the 6-month forward rates for the Treasury yield curve shown in Exhibit 4. The forward rates reported in Exhibit 12 are the annualized rates on a bond-equivalent basis. In Exhibit 13, the short-term forward rates are plotted along with the Treasury par yield curve and theoretical spot rate curve. The graph of the short-term forward rates is called the short-term forward-rate curve. Notice that the short-term forward rate curve lies above the other two curves. This will always be the case if the par yield curve is upward sloping. If the par yield curve is downward sloping, the short-term forward rate curve will be the lowest curve. Notice the unusual shape for the short-term forward rate curve. There is a mathematical reason for this shape. In practice, analysts will use statistical techniques to create a smooth short-term forward rate curve.

EXHIBIT 12 Six-Month Forward Rates (Annualized Rates on a Bond-Equivalent Basis)

Notation	Forward rate
$1f_0$	3.00
$1f_1$	3.60
$1f_2$	3.92
$1f_3$	5.15
$1f_4$	6.54
$1f_5$	6.33
$1f_6$	6.23
$1f_7$	5.79
$1f_8$	6.01
$1f_9$	6.24
$1f_{10}$	6.48
$1f_{11}$	6.72
$1f_{12}$	6.97
$1f_{13}$	6.36
$1f_{14}$	6.49
$1f_{15}$	6.62
$1f_{16}$	6.76
$1f_{17}$	8.10
$1f_{18}$	8.40
$1f_{19}$	8.71

EXHIBIT 13 Graph of Short-Term Forward Rate Curve



## Relationship between Spot Rates and Short-Term Forward Rates

Suppose an investor invests \$X in a 3-year zero-coupon Treasury security. The total proceeds three years (six periods) from now would be:

$$X(1 + z_6)^6$$

The investor could instead buy a 6-month Treasury bill and reinvest the proceeds every six months for three years. The future dollars or dollar return will depend on the 6-month forward rates. Suppose that the investor can actually reinvest the proceeds maturing every six months at the calculated 6-month forward rates shown in Exhibit 12. At the end of three years, an investment of X would generate the following proceeds:

$$X(1 + z_1)(1 + {}_1f_1)(1 + {}_1f_2)(1 + {}_1f_3)(1 + {}_1f_4)(1 + {}_1f_5)$$

Since the two investments must generate the same proceeds at the end of three years, the two previous equations can be equated:

$$X(1 + z_6)^6 = X(1 + z_1)(1 + {}_1f_1)(1 + {}_1f_2)(1 + {}_1f_3)(1 + {}_1f_4)(1 + {}_1f_5)$$

Solving for the 3-year (6-period) spot rate, we have:

$$z_6 = [(1 + z_1)(1 + {}_1f_1)(1 + {}_1f_2)(1 + {}_1f_3)(1 + {}_1f_4)(1 + {}_1f_5)]^{1/6} - 1$$

This equation tells us that the 3-year spot rate depends on the current 6-month spot rate and the five 6-month forward rates. In fact, the right-hand side of this equation is a geometric average of the current 6-month spot rate and the five 6-month forward rates.

Let's use the values in Exhibits 4 and 12 to confirm this result. Since the 6-month spot rate in Exhibit 4 is 3%,  $z_1$  is 1.5% and therefore:

$$\begin{aligned} z_6 &= [(1.015)(1.018)(1.0196)(1.0257)(1.0327)(1.03165)]^{1/6} - 1 \\ &= 0.023761 = 2.3761\% \end{aligned}$$

Doubling this rate gives 4.7522%. This agrees with the spot rate shown in Exhibit 4. In general, the relationship between a T-period spot rate, the current 6-month spot rate, and the 6-month forward rates is as follows:

$$z_T = [(1 + z_1)(1 + {}_1f_1)(1 + {}_1f_2) \dots (1 + {}_1f_{T-1})]^{1/T} - 1$$

Therefore, discounting at the forward rates will give the same present value as discounting at spot rates.

### **Valuation Using Forward Rates**

Since a spot rate is simply a package of short-term forward rates, it will not make any difference whether we discount cash flows using spot rates or forward rates. That is, suppose that the cash flow in period  $T$  is \$1. Then the present value of the cash flow can be found using the spot rate for period  $T$  as follows:

$$\text{PV of \$1 in } T \text{ periods} = \frac{1}{(1 + z_T)^T}$$

Alternatively, since we know that

$$z_T = [(1 + z_1)(1 + {}_1f_1)(1 + {}_1f_2) \cdots (1 + {}_1f_{T-1})]^{1/T} - 1$$

then, adding 1 to both sides of the equation,

$$(1 + z_T) = [(1 + z_1)(1 + {}_1f_1)(1 + {}_1f_2) \cdots (1 + {}_1f_{T-1})]^{1/T}$$

Raising both sides of the equation to the  $T$ -th power we get:

$$(1 + z_T)^T = (1 + z_1)(1 + {}_1f_1)(1 + {}_1f_2) \cdots (1 + {}_1f_{T-1})$$

Substituting the right-hand side of the above equation into the present value formula we get:

$$\text{PV of \$1 in } T \text{ periods} = \frac{1}{(1 + z_1)(1 + {}_1f_1)(1 + {}_1f_2) \cdots (1 + {}_1f_{T-1})}$$

In practice, the present value of \$1 in  $T$  periods is called the **forward discount factor for period  $T$** .

For example, consider the forward rates shown in Exhibit 12. The forward discount rate for period 4 is found as follows:

$$\begin{array}{ll} z_1 = 3\%/2 = 1.5\% & {}_1f_1 = 3.6\%/2 = 1.8\% \\ {}_1f_2 = 3.92\%/2 = 1.958\% & {}_1f_3 = 5.15\%/2 = 2.577\% \end{array}$$

$$\begin{aligned} \text{forward discount factor of \$1 in 4 periods} &= \frac{\$1}{(1.015)(1.018)(1.01958)(1.02577)} \\ &= 0.925369 \end{aligned}$$

To see that this is the same present value that would be obtained using the spot rates, note from Exhibit 4 that the 2-year spot rate is 3.9164%. Using that spot rate, we find:

$$\begin{aligned} z_4 &= 3.9164\%/2 = 1.9582\% \\ \text{PV of \$1 in 4 periods} &= \frac{\$1}{(1.019582)^4} = 0.925361 \end{aligned}$$

The answer is the same as the forward discount factor (the slight difference is due to rounding).

Exhibit 14 shows the computation of the forward discount factor for each period based on the forward rates in Exhibit 12. Let's show how both the forward rates and the spot rates can be used to value a 2-year 6% coupon Treasury bond. The present value for each cash flow is found as follows using spot rates:

$$\frac{\text{cash flow for period } t}{(1 + z_t)^t}$$

EXHIBIT 14 Calculation of the Forward Discount Factor for Each Period

Period	Years	Notation	Forward rate*	0.5 × Forward rate**	1 + Forward rate	Forward discount factor
1	0.5	${}_1f_0$	3.00%	1.5000%	1.01500	0.985222
2	1.0	${}_1f_1$	3.60%	1.8002%	1.01800	0.967799
3	1.5	${}_1f_2$	3.92%	1.9583%	1.01958	0.949211
4	2.0	${}_1f_3$	5.15%	2.5773%	1.02577	0.925362
5	2.5	${}_1f_4$	6.54%	3.2679%	1.03268	0.896079
6	3.0	${}_1f_5$	6.33%	3.1656%	1.03166	0.868582
7	3.5	${}_1f_6$	6.23%	3.1139%	1.03114	0.842352
8	4.0	${}_1f_7$	5.79%	2.8930%	1.02893	0.818668
9	4.5	${}_1f_8$	6.01%	3.0063%	1.03006	0.794775
10	5.0	${}_1f_9$	6.24%	3.1221%	1.03122	0.770712
11	5.5	${}_1f_{10}$	6.48%	3.2407%	1.03241	0.746520
12	6.0	${}_1f_{11}$	6.72%	3.3622%	1.03362	0.722237
13	6.5	${}_1f_{12}$	6.97%	3.4870%	1.03487	0.697901
14	7.0	${}_1f_{13}$	6.36%	3.1810%	1.03181	0.676385
15	7.5	${}_1f_{14}$	6.49%	3.2450%	1.03245	0.655126
16	8.0	${}_1f_{15}$	6.62%	3.3106%	1.03310	0.634132
17	8.5	${}_1f_{16}$	6.76%	3.3778%	1.03378	0.613412
18	9.0	${}_1f_{17}$	8.10%	4.0504%	1.04050	0.589534
19	9.5	${}_1f_{18}$	8.40%	4.2009%	1.04201	0.565767
20	10.0	${}_1f_{19}$	8.72%	4.3576%	1.04357	0.542142

\*The rates in this column are rounded to two decimal places.

\*\*The rates in this column used the forward rates in the previous column carried to four decimal places.

The following table uses the spot rates in Exhibit 4 to value this bond:

Period	Spot rate BEY (%)	Semiannual spot rate (%)	PV of \$1	Cash flow	PV of cash flow
1	3.0000	1.50000	0.9852217	3	2.955665
2	3.3000	1.65000	0.9677991	3	2.903397
3	3.5053	1.75266	0.9492109	3	2.847633
4	3.9164	1.95818	0.9253619	103	95.312278
				Total	104.018973

Based on the spot rates, the value of this bond is \$104.0190.

Using forward rates and the forward discount factors, the present value of the cash flow in period  $t$  is found as follows:

$$\text{cash flow in period } t \times \text{discount factor for period } t$$

The following table uses the forward rates and the forward discount factors in Exhibit 14 to value this bond:

Period	Semiann. forward rate	Forward discount factor	Cash flow	PV of cash flow
1	1.5000%	0.985222	3	2.955665
2	1.8002%	0.967799	3	2.903397
3	1.9583%	0.949211	3	2.847633
4	2.5773%	0.925362	103	95.312278
			Total	104.018973

The present value of this bond using forward rates is \$104.0190. So, it does not matter whether one discounts cash flows by spot rates or forward rates, the value is the same.

### Computing Any Forward Rate

Using spot rates, we can compute any forward rate. Using the same arbitrage arguments as used above to derive the 6-month forward rates, any forward rate can be obtained.

There are two elements to the forward rate. The first is when in the future the rate begins. The second is the length of time for the rate. For example, the 2-year forward rate 3 years from now means a rate three years from now for a length of two years. The notation used for a forward rate,  $f$ , will have two subscripts—one before  $f$  and one after  $f$  as shown below:

$${}_t f_m$$

The subscript before  $f$  is  $t$  and is the length of time that the rate applies. The subscript after  $f$  is  $m$  and is when the forward rate begins. That is,

the length of time of the forward rate  $f$  when the forward rate begins

Remember our time periods are still 6-month periods. Given the above notation, here is what the following mean:

Notation	Interpretation for the forward rate
${}_1f_{12}$	6-month (1-period) forward rate beginning 6 years (12 periods) from now
${}_2f_8$	1-year (2-period) forward rate beginning 4 years (8 periods) from now
${}_6f_4$	3-year (6-period) forward rate beginning 2 years (4 periods) from now
${}_8f_{10}$	4-year (8-period) forward rate beginning 5 years (10 periods) from now

To see how the formula for the forward rate is derived, consider the following two alternatives for an investor who wants to invest for  $m + t$  periods:

- buy a zero-coupon Treasury bond that matures in  $m + t$  periods, or
- buy a zero-coupon Treasury bond that matures in  $m$  periods and invest the proceeds at the maturity date in a zero-coupon Treasury bond that matures in  $t$  periods.

The investor will be indifferent between the two alternatives if they produce the same return over the  $m + t$  investment horizon.

For \$100 invested in the first alternative, the proceeds for this investment at the horizon date assuming that the semiannual rate is  $z_{m+t}$  is

$$\$100 (1 + z_{m+t})^{m+t}$$

For the second alternative, the proceeds for this investment at the end of  $m$  periods assuming that the semiannual rate is  $z_m$  is

$$\$100 (1 + z_m)^m$$

When the proceeds are received in  $m$  periods, they are reinvested at the forward rate,  ${}_t f_m$ , producing a value for the investment at the end of  $m + t$  periods of

$$\$100 (1 + z_m)^m (1 + {}_t f_m)^t$$

For the investor to be indifferent to the two alternatives, the following relationship must hold:

$$\$100 (1 + z_{m+t})^{m+t} = \$100 (1 + z_m)^m (1 + {}_t f_m)^t$$

Solving for  ${}_t f_m$  we get:

$${}_t f_m = \left[ \frac{(1 + z_{m+t})^{m+t}}{(1 + z_m)^m} \right]^{1/t} - 1$$

Notice that if  $t$  is equal to 1, the formula reduces to the 1-period (6-month) forward rate.

To illustrate, for the spot rates shown in Exhibit 4, suppose that an investor wants to know the 2-year forward rate three years from now. In terms of the notation,  $t$  is equal to 4 and  $m$  is equal to 6. Substituting for  $t$  and  $m$  into the equation for the forward rate we have:

$${}_4 f_6 = \left[ \frac{(1 + z_{10})^{10}}{(1 + z_6)^6} \right]^{1/4} - 1$$

This means that the following two spot rates are needed:  $z_6$  (the 3-year spot rate) and  $z_{10}$  (the 5-year spot rate). From Exhibit 4 we know

$$z_6(\text{the 3-year spot rate}) = 4.752\%/2 = 0.02376$$

$$z_{10}(\text{the 5-year spot rate}) = 5.2772\%/2 = 0.026386$$

then

$${}_4 f_6 = \left[ \frac{(1.026386)^{10}}{(1.02376)^6} \right]^{1/4} - 1 = 0.030338$$

Therefore,  ${}_4 f_6$  is equal to 3.0338% and doubling this rate gives 6.0675% the forward rate on a bond-equivalent basis.

We can verify this result. Investing \$100 for 10 periods at the spot rate of 2.6386% will produce the following value:

$$\$100 (1.026386)^{10} = \$129.7499$$

Investing \$100 for 6 periods at 2.376% and reinvesting the proceeds for 4 periods at the forward rate of 3.030338% gives the same value:

$$\$100 (1.02376)^6 (1.030338)^4 = \$129.75012$$