Financial Securities (2UZB605)

Week 6:

- Binomial trees
- Black-Scholes model

One-step binomial model

Call option with a strike price of $11 and 12 months to expiry

Assumptions:

1. **No arbitrate opportunity**
2. **There are only 2 outcomes for stock price $12 or $8**
3. **Risk free rate = 8%**

Action:

1. **Buy stock at $10**
2. **Issue (i.e. short) 1 call option**

Stock price = $10

Stock price = $12
Max(0, $12 - $11) = 1

Stock price = $8
Max(0, $8 - $11) = 0

Then find the number of shares required to make the outcome the SAME whether the stock price goes to $12 or $8

Option pricing basics

$u > 1$, u – up movement in stock price (1 + % increase)
$d < 1$, d – down movement in stock price (1 - % decrease)

f(u) – option value when the stock price is up
f(d) – option value when the stock price is down

If up movement the value of the portfolio is: $S(0)*u*\Delta - f(u)$
If down movement the value of the portfolio is: $S(0)*d*\Delta - f(d)$
Since the portfolio is riskless with a short call: $S(0)*u*\Delta - f(u) = S(0)*d*\Delta - f(d)$
$\Delta = [f(u) – f(d)] / [S(0)*u – S(0)*d)]$

Riskless portfolio earns *risk free rate.* Therefore, the present of the portfolio will be:
$[[S(0)*u*\Delta - f(u)]*e^{(-rT)}$

The cost of setting up the portfolio must equal the present value: $S(0)*\Delta - f = [[S(0)*u*\Delta - f(u)]*e^{(-rT)}$

Thus, $f = S(0)*\Delta*(1 - u*e^{(-rT)}) + f(u)*e^{(-rT)}$

$f = e^{(-rT)}*[p*f(u) + (1-p)*f(d)]$
$p = (e^{(-rT)} – d) / (u – d)$

Option pricing basics

Let's say
u = 1.1, d = 0.9, r = 0.08, T = 1, f(u) = 1, f(d) = 0

p = [e^(0.08 * 1) – 0.9] / [1.1 – 0.9] = 0.0231 / 0.2 = 0.116

f = e^(-0.08)*[0.116*1 + (1-0.116)*0] = 0.107

Black-Scholes Option Pricing

Ln – natural logarithm (used to accommodate account for continuously compounded interest)
C – call option price
S – current asset price
K – strike price
r – short-term risk-free rate
e – 2.718
(T – t) – time to expiry
σ – sigma or standard deviation of asset return
N(..) – cumulative probability density function (the value of N(..) is obtained from a **normal distribution** function that is tabulated in most statistics textbooks)

$$C = N(d_1) \cdot S - N(d_2) \cdot Ke^{-r(T-t)}$$

$$P = C + Ke^{-r(T-t)} - S$$

$$d_1 = \frac{\ln\left(\frac{S}{K}\right) + \left(r + \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}}$$

$$d_2 = d_1 - \sigma\sqrt{T-t}$$

Black-Scholes Option Pricing

Example:

Zero-coupon bond with 3 years to maturity.
K = $88
Time to expiry (T-t) = 3 years
Current price = $83.96
Expected return volatility(σ "sigma") = 10% or 0.1
Risk-free rate = 6%
d1 = [ln(83.96/88) + (0.06 + (0.1^2)/2)*2] / [0.1*SQRT(2)]
d2 = d1 – σ*SQRT(2)

$$C = N(d_1) \cdot S - N(d_2) \cdot Ke^{-r(T-t)}$$

$$P = C + Ke^{-r(T-t)} - S$$

$$d_1 = \frac{\ln\left(\frac{S}{K}\right) + \left(r + \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}}$$

$$d_2 = d_1 - \sigma\sqrt{T-t}$$

Recommended reading for Introduction to Derivatives:

Fabozzi, Bond Markets, Analysis and Strategies, Chapters 21, 22 and 23 (Interest rate futures, Interest rate options and Interest rate swaps

Fabozzi, Fixed Income Analysis, Chapters 13 and 14 (Interest rate derivatives and Valuation of interest rate derivatives)

Choudhry (Introduction to Bond Markets): Chapter 11, Introduction to derivative instruments

Chance & Brooks, Introduction to Derivatives and Risk Management, Chapters 2, 3, 8, 9, 12, 13